

Universal behavior of the coefficients of the continuous equation in competitive growth models

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The competitive growth models (CGM) involving only one kind of particles, are a mixture of two processes, one with probability p and the other with probability $1-p$. The p dependance produce crossovers between two different regimes. We demonstrate that the coefficients of the continuous equation, describing their universality classes, are quadratic in p (or $1-p$). We show that the origin of such dependance is the existence of two different average time rates. Thus, the quadratic p dependance is a universal behavior of all the (CGM). We derive analytically the continuous equations for two CGM, in $1+1$ dimensions, from the microscopic rules using a regularization procedure. We propose generalized scalings that reproduce the scaling behavior in each regime. In order to verify the analytic results and the scalings, we perform numerical integrations of the derived analytical equations. The results are in excellent agreement with those of the microscopic CGM presented here and with the proposed scalings.

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Evolving growing interfaces or surfaces can be found in many physical, chemical and biological processes. For example, in film growth either by vapor deposition or chemical deposition [1,2], bacterial growth [3] and propagation of forest fire [4]. The resulting interface has a rough surface that is characterized through scaling of the interfacial width W defined as $W(L,t) = \{\langle [h_i - \langle h_i \rangle]^2 \rangle^{1/2}\}$, where h_i is the height at the position i , $\langle h_i \rangle = \sum_{i=1}^L h_i$ is the spatial average, L is the linear size, d is the spatial dimension and $\{\}$ denote configuration averages. The general scaling relation [1] for these growing interfaces that evolves through a single model can be summarized in the form $W(L,t) \sim L^\alpha f(t/L^z)$, where the scaling function $f(u)$ behaves as $f(u) \sim u^\beta$ ($\beta = z/\alpha$), for $u \ll 1$ and $f(u) \sim \text{const}$ for $u \gg 1$. The exponent α describes the asymptotic behavior where the width saturates due to finite sizes effects, while the exponent β represents the early time regime where finite-size effects are weak. The crossover time between the two regimes is $t_s = L^z$.

The study of growth models involving one kind of particle in competitive processes has received little attention, in spite of the fact that they are more realistic describing the growing in real materials, where usually there exists a competition between different growing processes. As an example, in a colony of bacteria growing on a substrate, a new bacteria can be borne near to another and stay there, move into another place looking for food or die. This “bacteria” can be thought as a particle undergoing either a deposition/evaporation process or deposition/surface relaxation.

The processes involved in the competitive growth models (CGM) could have different characteristic average time rate. Recently Shapir *et al.* [5] reported experimental results of surface roughening during cyclical electrodeposition dissolution of silver. Horowitz, Monetti, and Albano [6] introduced a competitive growth model between random deposition with surface relaxation (RDSR) with probability p and random deposition (RD) with probability $1-p$, called RDSR/RD. The authors proposed that the scaling behavior is character-

istic of an Edward Wilkinson (EW) equation, where the coefficient associated with the surface tension ν depends on p . The dependance of ν on p governs the transition from RDSR to RD. Using a dynamic scaling ansatz for the interface width W they found that the results are consistent provided that $\nu \propto p^2$. Also Pellegrini and Jullien [7] have introduced CGM between ballistic deposition (BD) with probability p and RDSR with probability $1-p$, called BD/RDSR. For this model Chame and Aarão Reis [8] presented a more careful analysis in $1+1-d$ and showed that there exists a slow crossover from an EW to a Kardar-Parisi-Zhang (KPZ) for any $p > 0$. They also found that the parameter p is connected to the coefficient λ of the nonlinear term of the KPZ equation by $\lambda \sim p^\gamma$, with $\gamma = 2.1$.

In this communication, we show that the origin of such dependance is the existence of two different average time rates. Thus, the quadratic p dependance is a universal feature of all the CGM. The p dependance on the coefficients of the continuous equations is obtained analytically from the microscopic dynamics.

In order to test our hypothesis, we derive the analytical continuous equations for the local height for the RDSR/RD and BD/RDSR models. The procedure chosen here is based on regularization and coarse graining of the discrete Langevin equations obtained from a Kramers-Moyal expansion of the master equation [9–11].

Let us introduce first the general treatment of this problem. Let us denote by $h_i(t)$ the height of the i th generic site at time t . The set $\{h_i, i = 1, \dots, L\}$ defines the interface. Here we distinguish between two competitive processes: A with probability p and average time of deposition τ_A , and B with probability $1-p$ and average time of deposition τ_B . In deposition processes with $p=1$ the average time of deposition τ_0 is given by $\tau_0^{-1} = \{dh_i/dt\}_{(p=1)}$. If the process is made with probability p then $\tau_A^{-1} = \{dh_i/dt\}_{(p)} = p \{dh_i/dt\}_{(p=1)}$. The same holds for a process with probability $(1-p)$. Thus, the particles are deposited at an average rate

$$\tau_A = \frac{\tau_0}{p}, \quad \tau_B = \frac{\tau_0}{1-p}. \quad (1)$$

In the average time of each process, the height in the site i increases by

$$h_i(t + \tau_A) = h_i(t) + a_{\perp} p R_i^A, \quad (2)$$

$$h_i(t + \tau_B) = h_i(t) + a_{\perp} (1-p) R_i^B,$$

where R_i^A and R_i^B are the growing rules for processes A and B , respectively, and a_{\perp} is the vertical lattice spacing. Expanding $h_i(t + \tau_A)$ and $h_i(t + \tau_B)$ to second order in Taylor series around τ_A and τ_B , we obtain

$$h_i(t + \tau_J) - h_i(t) \approx \frac{dh_i}{dt} \tau_J, \quad (3)$$

for the process $J=A, B$. Thus, the evolution equation for the height (in the site i) for this CGM is given by

$$\frac{dh_i}{dt} = K_i^{(1,A)} + K_i^{(1,B)} + \eta_i(t), \quad (4)$$

where the first moments of the transition rate for each process [12,13] are

$$K_i^{(1,A)} = \frac{a_{\perp}}{\tau_A} p R_i^A,$$

$$K_i^{(1,B)} = \frac{a_{\perp}}{\tau_B} (1-p) R_i^B, \quad (5)$$

and the Gaussian thermal noise $\eta_i(t)$ has zero mean and covariance

$$\{\eta_i(t) \eta_j(t')\} = a_{\perp} (K_i^{(1,A)} + K_i^{(1,B)}) \delta_{ij} \delta(t - t'). \quad (6)$$

In order to test our analytical result, we use two models. The first model RDSR/RD [6] considers a mixture of RDSR (process A) with probability p and RD (process B) with probability $1-p$. Let us introduce the growth rule for each process for the first model. In the RD growth model one chose a column of a lattice, at random, among L and a particle is launched until it reaches the top of the selected column. The RDSR is a variant of the RD: a particle is released from a random position but when it reaches the top of the selected column is allowed to relax to the lowest nearest neighbor (nn) column if at least one of the nn heights is lower than the selected one. If the height of both of the nn is lower than the selected one the relaxation takes place with equal probability to one of them. For RD, $W(L, t)$ does not depend on L . This means that the width W does not saturate due to the lack of lateral correlations. Thus, in this model: $W(t) \sim t^{\beta_{RD}}$. Moreover, the RDSR model generates lateral correlations, therefore one has $\beta_{RDSR} = 1/4$ and $\alpha_{RDSR} = 1/2$. The first moment of the transition rate for these processes is

$$K_i^{(1,A)} = \frac{a_{\perp}}{\tau_A} p (\omega_i^{(2)} + \omega_{i+1}^{(3)} + \omega_{i-1}^{(4)}),$$

$$K_i^{(1,B)} = \frac{a_{\perp}}{\tau_B} (1-p) \omega_i^{(1)}, \quad (7)$$

where the rules for both processes can we written as

$$\omega_i^{(1)} = 1,$$

$$\omega_i^{(2)} = \Theta(H_i^{i+1}) \Theta(H_i^{i-1}),$$

$$\omega_i^{(3)} = \left\{ \frac{1}{2} [1 - \Theta(H_i^{i+1})] + \Theta(H_i^{i+1}) \right\} [1 - \Theta(H_i^{i-1})],$$

$$\omega_i^{(4)} = \left\{ \frac{1}{2} [1 - \Theta(H_i^{i-1})] + \Theta(H_i^{i-1}) \right\} [1 - \Theta(H_i^{i+1})], \quad (8)$$

where $H_{i\pm k}^{\pm s} = (h_{i\pm s} - h_{i\pm k}) / a_{\perp}$, and $\Theta(z)$ is the unit step function defined as $\Theta(z) = 1$ for $z \geq 0$ and $\Theta(z) = 0$ for $z < 0$. The representation of the step function can be expanded as $\Theta(z) = \sum_{k=0}^{\infty} c_k z^k$ providing that z is smooth. In any discrete model there is in principle an infinite number of nonlinearities, but at long wavelengths the higher order derivatives can be neglected using scaling arguments, since one expects affine interfaces over a long range of scales, and then one is usually concerned with the form of the relevant terms. Thus, keeping the expansion of the step function to first order in his argument and replacing the expansion Eq. (8), Eq. (4) can be written as

$$\frac{dh_i}{dt} = \frac{a_{\perp} (1-p)}{\tau_B} + \frac{a_{\perp} p}{\tau_A} \left(1 + c_1 \frac{\Delta^2 h_i}{a_{\perp}} \right) + \eta_i(t), \quad (9)$$

where $\Delta^2 h_i = h_{i+1} - 2h_i + h_{i-1} \approx a_{\parallel}^2 \partial^2 h / \partial x^2|_{h_i}$, and a_{\parallel} is the horizontal lattice spacing. Replacing the rates given by Eq. (1) in Eq. (9) and using a standard coarse-grain approach [10,11] the continuous equation for this CGM is

$$\frac{dh}{dt} = F(p) + \nu(p) \frac{\partial^2 h}{\partial x^2} + \eta(x, t), \quad (10)$$

where $h = h(x, t)$ and

$$F(p) = \frac{a_{\perp}}{\tau_0} [(1-p)^2 + p^2],$$

$$\nu(p) = 2 c_1 \frac{a_{\parallel}^2}{\tau_0} p^2. \quad (11)$$

The noise covariance is given by

$$\{\eta(x, t) \eta(x', t')\} = D(p) \delta(x - x') \delta(t - t'), \quad (12)$$

where

$$D(p) = a_{\parallel} a_{\perp} F(p). \quad (13)$$

Equations (11) and (13) show that the quadratic dependance on the coefficients of the continuous equation, arises naturally as a feature of the CGM and is due to the existence of different average time rates.

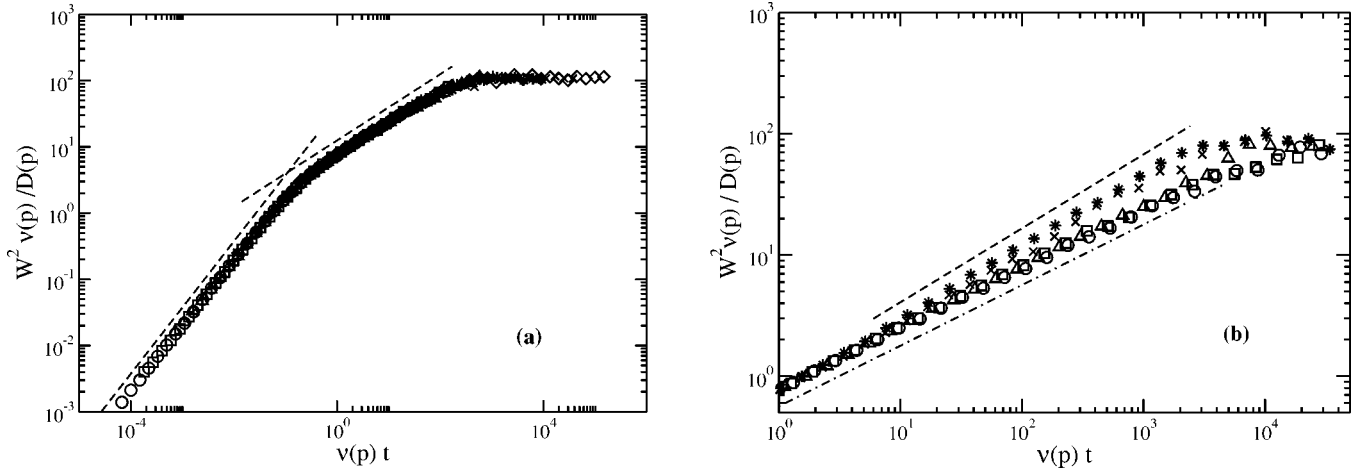


FIG. 1. (a) Log-log plot of $W^2 \nu(p)/D(p)$ for the RDSR/RD model as function of $\nu(p)t$ for $L=128$. The different symbols represent different values of p , $p=0.04$ (\circ), $p=0.08$ (\square), $p=0.016$ (\triangle), $p=0.32$ ($+$), and $p=0.64$ ($*$). Here we used $C=2.58$ and $b=0.2$ as parameters of the Θ -function representation. The dashed lines are used as guides to show the RD regime with $2\beta=1$ and the EW regime with $2\beta=0.5$. (b) Log-log plot of $W^2 \nu(p)/D(p)$ for the BD/RDSR model as function of $\nu(p)t$ for $L=1024$. The symbols represent the same as in Fig. 1(a). Here we used $C=0.18$ and $b=0.5$. The collapse of the curves at the earlier stage clearly shows the EW behavior ($2\beta=0.5$). After this stage the curves split and undergo a slow crossover to the KPZ behavior ($2\beta=0.66$). The dashed lines are used as guides to show the EW regime with $2\beta=1/2$ and the KPZ regime. The slope showed here is $2\beta=0.61$.

The second model is a mixture of RDSR with probability $1-p$ (B process) and ballistic deposition (BD) with probability p (A process) [8]. The evolution rules for RDSR are ω_i^j , with $j=2,3,4$ [see Eq. (8)]. In the BD model, the incident particle follows a straight trajectory and sticks to the surface at the column i . The height in the column i is increased in $\max[h_i+1, h_{i+1}, h_{i-1}]$. If this process is done with probability p (A process), the rules can be summarized as

$$\begin{aligned}\omega_i^{(5)} &= \Theta(H_{i+1}^i) \Theta(H_{i-1}^i), \\ \omega_i^{(6)} &= H_{i+1}^{i+1} [1 - \Theta(H_{i+1}^i)] [1 - \Theta(H_{i+1}^{i-1})], \\ \omega_i^{(7)} &= H_{i-1}^{i-1} [1 - \Theta(H_{i-1}^i)] [1 - \Theta(H_{i-1}^{i+1})],\end{aligned}\quad (14)$$

$$\omega_i^{(8)} = \frac{1}{2} \delta(H_{i-1}^{i+1}, 0) \{H_{i+1}^{i+1} [1 - \Theta(H_{i+1}^i)] + H_{i-1}^{i-1} [1 - \Theta(H_{i-1}^i)]\},$$

where $\delta(z, 0) = \Theta(z) + \Theta(-z) - 1$ is the Kronecker delta. Following the steps leading to Eq. (10) the evolution equation for this process can be written as

$$\frac{dh}{dt} = F(p) + \nu(p) \frac{\partial^2 h}{\partial x^2} + \lambda(p) \left(\frac{\partial h}{\partial x} \right)^2 + \eta(x, t), \quad (15)$$

where

$$F(p) = \frac{a_{\perp}}{\tau_0} [(1-p)^2 + c_0^2 p^2],$$

$$\nu(p) = \frac{a_{\parallel}^2}{\tau_0} \left[\frac{1}{2} p^2 (1 - c_0 - 2c_0 c_1) + 2c_1 (1-p)^2 \right], \quad (16)$$

$$\lambda(p) = \frac{a_{\parallel}^2}{\tau_0 a_{\perp}} p^2 c_1 (5 - 4c_0 - c_1).$$

The covariance of noise and $D(p)$ is given by Eq. (12) and (13), respectively. Notice that we have to change p by $1-p$ in all the above equations for RDSR, because in the first model RDSR is a kind A process and now is a kind B process. Equation (16) shows again the quadratic dependance on the coefficients of the continuous equation. The quadratic dependance of λ on p , found by Chame and Aarão Reis [8], is a general feature of the CGM.

As both models have an EW behavior, it is expected that in that regime the following generalized scaling ansatz [6,14]

$$W^2(p, L, t) \sim L^{2\alpha} [D(p)/\nu(p)] f(\nu(p)t/L^z), \quad (17)$$

where $f(u) \sim u^{2\beta}$ for $u \ll 1$ and $f(u) \sim \text{const}$ for $u \gg 1$. Moreover, the second model is represented by a mixture of EW and KPZ universality classes. In the early time regime $W(t) \sim t^{\beta_{\text{RDSR}}}$, while a crossover to a KPZ, with $\beta_{\text{KPZ}}=1/3$ and $\alpha \equiv \alpha_{\text{KPZ}}=1/2$, is expected in the intermediate regime before the saturation. Thus, for the KPZ regime we propose the following generalization [15] of the scaling behavior of the width:

$$W^2(p, L, t) \sim L^{2\alpha} [D(p)/\nu(p)] f(\lambda(p) \sqrt{D(p)/\nu(p)} t/L^z), \quad (18)$$

where $z=3/2$, and $f(u) \sim u^{2\beta_{\text{KPZ}}}$ for $u \ll 1$ and $f(u) \sim \text{const}$ for $u \gg 1$.

In order to test our analytical result and the proposed scalings, we perform a numerical integration of Eq. (10) and (15), and compute W^2 for both models.

Notice that in order to numerically integrate the continuous equation, we do need a continuum representation of the Θ function to numerically compute the coefficients c_0 and c_1

related to the ones of the continuous equations. To perform the numerical integration, we chose the shifted hyperbolic tangent [16] as the continuous representation of Θ function defined as $\Theta(x)=\{1+\tanh[C(x+b)]\}/2$, where b is the shift and C is a parameter that allows one to recover the Θ in the limit $C\rightarrow\infty$. The numerical integration was made in short lattices using a discretized version of the continuous Eqs. (10) and (15). The results in large systems and the details of the integration are beyond the scope of this communication and will be published elsewhere.

For the first model, Horowitz, Monetti, and Albano [6] presented their data from simulations plotting the scaling relation $W/L^\alpha p^{-\delta}$ vs $t/L^\zeta p^{-\gamma}$. Clearly, their $\delta=1$ and $\gamma=2$ is related to our $\nu(p)$ and $D(p)$ [see Eq. (11) and (13)]. In Fig. 1(a) we plot $W^2 \nu(p)/D(p)$ as function of $\nu(p)t$ for that model for different values of p and $L=128$. This figure represent the same as in [6] after coarse graining. The agreement with the results of our numerical integration, the numerical simulation [6] and the scaling presented in Eq. (17), is excellent. On the other hand, for the second model, Chame and Aarão Reis [8] did not present the result for W . They studied the crossover from EW to KPZ using an indirect method because of the slow convergence of the discrete model to KPZ behavior. The crossover is well represented in our Fig. 1(b), where we plot the same as in Fig. 1(a) but for the second model. The collapse of the curves in the EW regime is clear. In the intermediate regime the KPZ behavior appears thus, and it is expected that Eq. (18) holds in that regime. In Fig. 2 we plot $\tilde{W}^2=W^2 \nu(p)/[L^{2\alpha}D(p)]$ as function of $\tilde{t}=\lambda(p)\sqrt{D(p)/\nu(p)} t/L^\zeta$ for three different values of p using $z=3/2$. As p increases, the KPZ behavior appears earlier, but independent of p all the curves saturate as a KPZ. The agreement with Eq. (18) is excellent in the saturation regime. The departure in the intermediate regime is due to a slow crossover to the KPZ and to finite size effects.

Finally, notice that the quadratic dependance of the coefficients of the continuous equation on p is independent of the

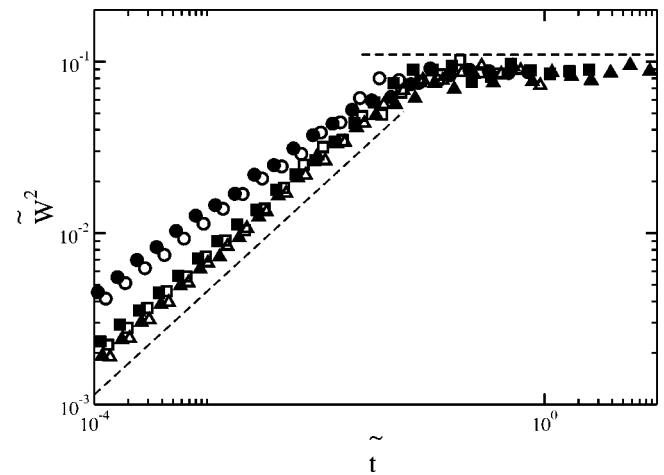


FIG. 2. Log-log plot of \tilde{W}^2 as function of \tilde{t} as defined in the text, for $p=0.16$ (○), $p=0.32$ (□) and $p=0.64$ (△). The empty symbols correspond to $L=512$ and the filled ones to $L=1024$. The collapse of the curves on the saturation regime using $z=3/2$ shows that the curves saturates with a KPZ behavior as expected.

CGM considered, because it is due to two different rates of deposition given by Eq. (1). This dependance is totally general, as shown from Eq. (1) to Eq. (7).

In summary, we demonstrate that the coefficients of the continuous equation have quadratic dependance on p . This feature is universal for all the CGM and is due to the competition between different average time rates. We propose generalized scaling for the model that reproduces the scaling behavior in each regime. The numerical integration of the continuous equation are in excellent agreement with the propose scalings and the numerical simulation of the models.

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